AP Calculus AB

Summer Assignment

and

Reference Sheet

This packet has information to help you prepare for Calculus next fall. There is a short introduction to "What is Calculus?", followed by information from Pre-Calculus (all math prior to Calculus is referred to as Pre-Calculus) which you will be expected to *know*. Finally, there is an assignment that will be due the first day of class next fall for you to complete over the summer. I will post the answers on my website, the end of July so you can verify your work.

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What Is Calculus?

Calculus is a part of mathematics that evolved much later than other subjects. Algebra, geometry, and trigonometry were developed in ancient times, but calculus as we know it did not appear until the seventeenth century.

The first evidence of calculus has its roots in ancient mathematics. For example, In his book *A History of* π, Petr Beckmann explains that Greek mathematician Archimedes (287–212 bce) "took the step from the concept of 'equal to' to the concept of 'arbitrarily close to' or 'as closely as desired'...and reached the threshold of the differential calculus, just as his method of squaring the parabola reached the threshold of the integral calculus."[1] But it was not until Sir Isaac Newton and Gottfried Wilhelm Leibniz, each working independently, expanded, organized, and applied these early ideas, that the subject we now know as calculus was born.

Although we attribute the birth of calculus to Newton and Leibniz, many other mathematicians, particularly those in the eighteenth and nineteenth centuries, contributed greatly to the body and rigor of calculus. You will encounter many of their names and contributions as you pursue your study of calculus.

But what is calculus?

The simple answer is: calculus models change. Since the world and most things in it are constantly changing, mathematics that explains change becomes immensely useful.

Calculus has two major branches, differential calculus and integral calculus. Let's take a peek at what calculus is by looking at two problems that prompted the development of calculus.

The Tangent Problem—The Basis of Differential Calculus

Suppose we want to find the slope of the line tangent to the graph of a function at some point $P = (x_1, y_1)$. See Figure 1(a). Since the tangent line necessarily contains the point P, it remains only to find the slope to identify the tangent line. Suppose we repeatedly zoom in on the graph of the function at the point P. See Figure 1(b). If we can zoom in close enough, then the graph of the function will look approximately linear, and we can choose a point Q, on the graph of the function different from the point P, and use the formula for slope.

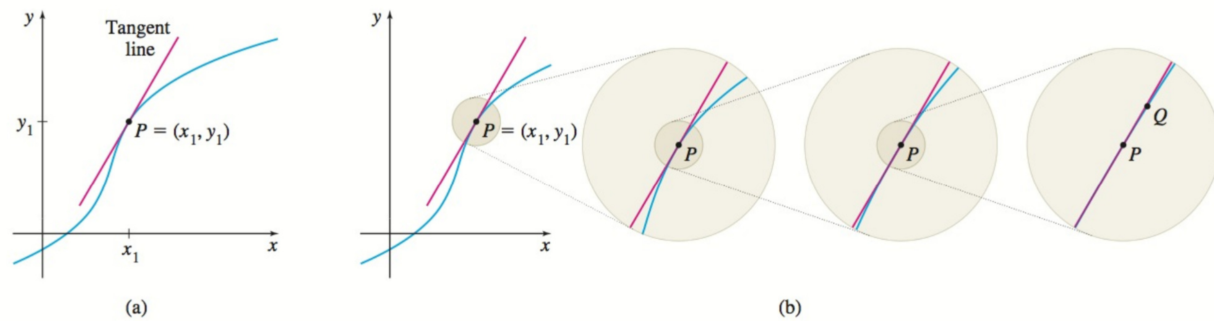


Figure 1

Repeatedly zooming in on the point P is equivalent to choosing a point Q closer and closer to the point P. Notice that as we zoom in on P, the line connecting the points P and Q, called a secant line, begins to look more and more like the tangent line to the graph of the function at the point P. If the point Q can be made as close as we please to the point P, without equaling the point P, then the slope of the tangent line m_{tan} can be found. This formulation leads to differential calculus, the study of the derivative of a function.

[1] Beckmann, P. (1976). A History of π (3rd. ed., p. 64). New York: St. Martin's Press.

The derivative gives us information about how a function changes at a given instant and can be used to solve problems involving velocity and acceleration; marginal cost and profit; and the rate of change of a chemical reaction. Derivatives are the subjects of Chapters 2 through 4.

The Area Problem—The Basis of Integral Calculus

If we want to find the area of a rectangle or the area of a circle, formulas are available. (See Figure 2.) But what if the figure is curvy, but not circular as in Figure 3? How do we find this area?

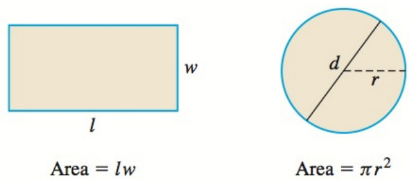


Figure 2

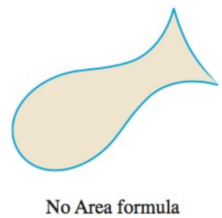


Figure 3

Calculus provides a way. Look at Figure 4(a). It shows the graph of $y = x^2$ from $x = 0$ to $x = 1$. Suppose we want to find the area of the shaded region.

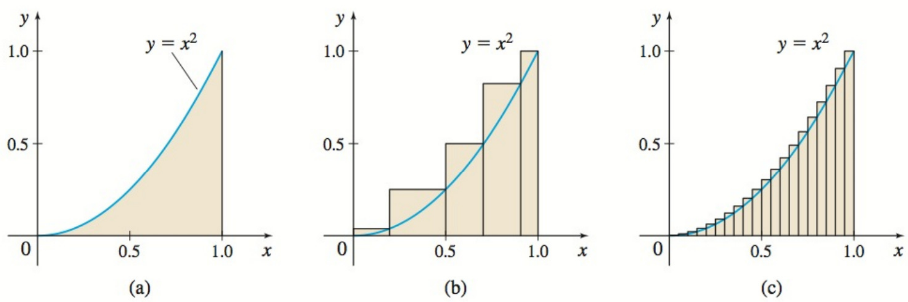


Figure 4

By subdividing the x-axis from 0 to 1 into small segments and drawing a rectangle of height x^2 above each segment, as in Figure 4(b), we can find the area of each rectangle and add them together. This sum approximates the shaded area in Figure 4(a). The smaller we make the segments of the x-axis and the more rectangles we draw, the better the approximation becomes. See Figure 4(c). This formulation leads to integral calculus, and the study of the integral of a function.

Two Problems—One Subject?

At first differential calculus (the tangent problem) and integral calculus (the area problem) appear to be different, so why call both of them calculus? The Fundamental Theorem of Calculus establishes that the derivative and the integral are related. In fact, one of Newton's teachers, Isaac Barrow, recognized that the tangent problem and the area problem are closely related, and that derivatives and integrals are inverses of each other. Both Newton and Leibniz formalized this relationship between derivatives and integrals in the *Fundamental Theorem of Calculus*.

Algebra

Factorization & Expansion

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 + b^3$$
$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b)(a^2 - ab + b^2) = a^3 + b^3$$
$$(a-b)(a^2 + ab + b^2) = a^3 - b^3$$

Equations & Formulas

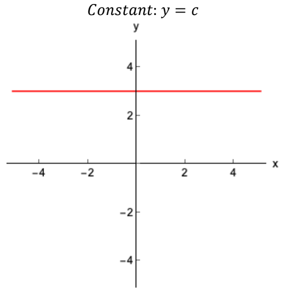
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$y = mx + b$$
$$y - y_1 = m(x - x_1)$$
$$y = ax^2 + bx + c$$
$$A = P\left(1 \pm \frac{r}{n}\right)^{nt}$$
$$A = Pe^{rt}$$
$$a^2 + b^2 = c^2$$

Properties of Logarithms

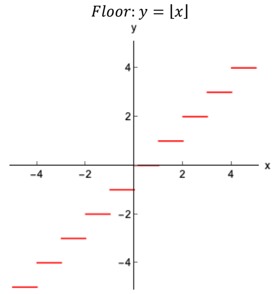
$$y = a^x \leftrightarrow \log_a y = x$$
$$y = 10^x \leftrightarrow \log y = x$$
$$y = e^x \leftrightarrow \ln y = x$$
$$\log_a(xy) = \log_a(x) + \log_a(y)$$
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$
$$\log_a x^n = n \log_a x$$

Basic Parent Functions and Their Graphs

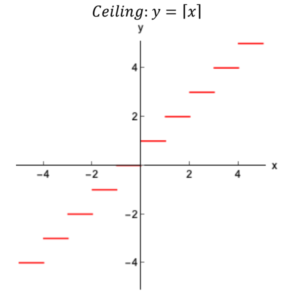
Constant: $y = c$



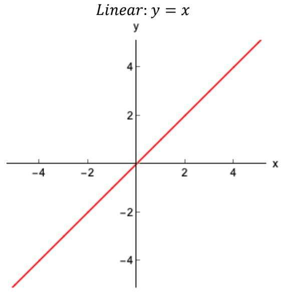
Floor: $y = \lfloor x \rfloor$



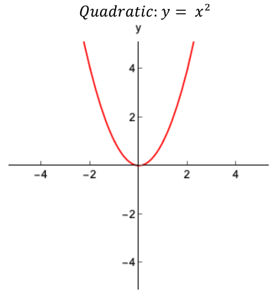
Ceiling: $y = \lceil x \rceil$



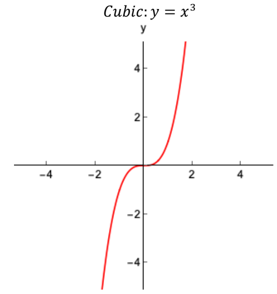
Linear: $y = x$



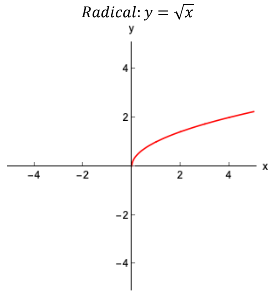
Quadratic: $y = x^2$



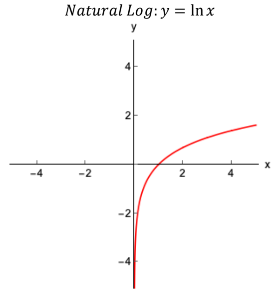
Cubic: $y = x^3$



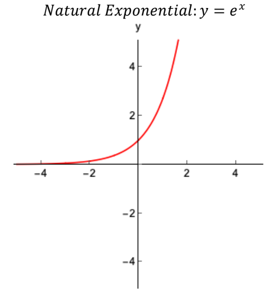
Radical: $y = \sqrt{x}$



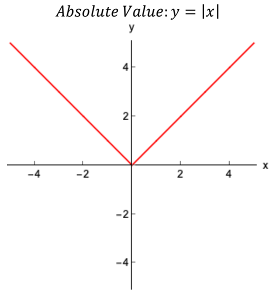
Natural Log: $y = \ln x$



Natural Exponential: $y = e^x$



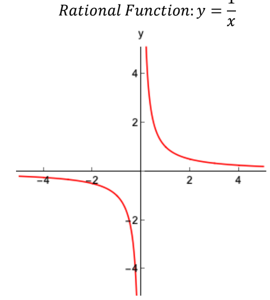
Absolute Value: $y = |x|$



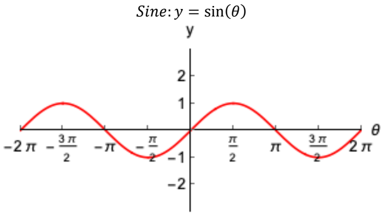
Functional Notations

$$y = f(x)$$
$$(f \circ g)x = f(g(x))$$
$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

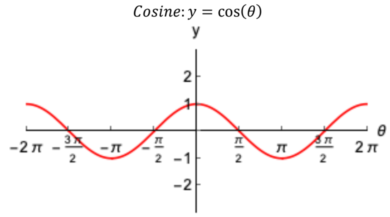
Rational Function: $y = \frac{1}{x}$



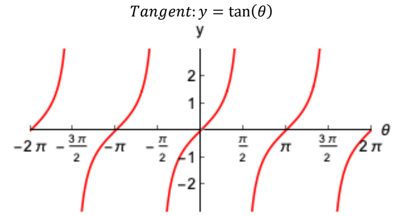
Sine: $y = \sin(\theta)$



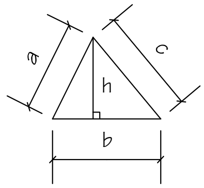
Cosine: $y = \cos(\theta)$



Tangent: $y = \tan(\theta)$

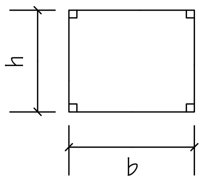


Geometry



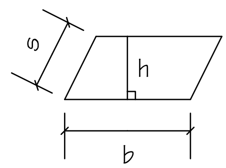
$P = a + b + c$

$A = \frac{1}{2}bh$



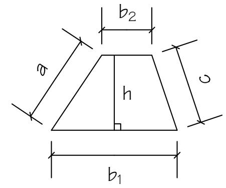
$P = 2h + 2b$

$A = bh$



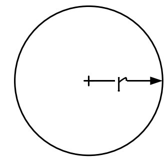
$P = 2s + 2b$

$A = bh$



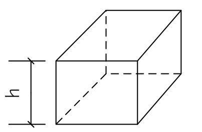
$P = a + c + b_1 + b_2$

$A = \frac{1}{2}h(b_1 + b_2)$



$C = \pi d = 2\pi r$

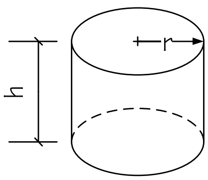
$A = \pi r^2$



B = area of the base

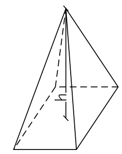
$SA = \sum(\text{Face Areas})$

$V = Bh$



$SA = 2\pi rh + 2\pi r^2$

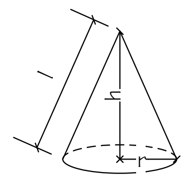
$V = \pi r^2 h$



B=area of the base

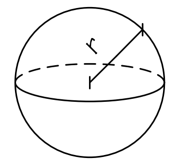
$SA = \sum(\text{Face Areas})$

$V = \frac{1}{3}Bh$



$SA = \pi r^2 + \pi r l$

$V = \frac{1}{3}\pi r^2 h$

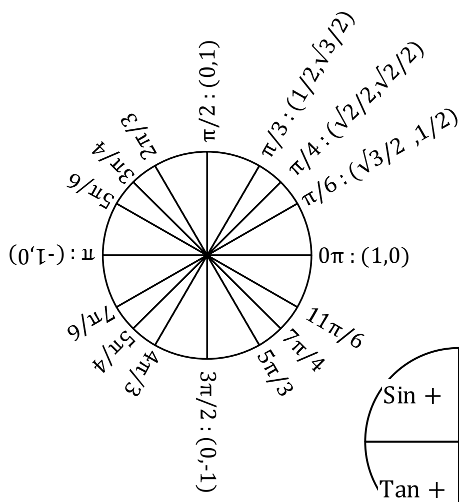


$SA = 4\pi r^2$

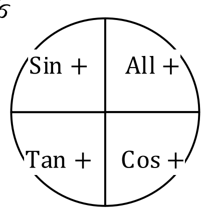
$V = \frac{4}{3}\pi r^3$

Trigonometry

Unit Circle



radians: (cos(α),sin(α))



Ratio Definitions

$\sin(\varphi) = \frac{\text{Opp}}{\text{Hyp}}$

$\cos(\varphi) = \frac{\text{Adj}}{\text{Hyp}}$

$\tan(\varphi) = \frac{\text{Opp}}{\text{Adj}} = \frac{\sin(\varphi)}{\cos(\varphi)}$

$\csc(\varphi) = \frac{\text{Hyp}}{\text{Opp}} = \frac{1}{\sin(\varphi)}$

$\sec(\varphi) = \frac{\text{Hyp}}{\text{Adj}} = \frac{1}{\cos(\varphi)}$

$\cot(\varphi) = \frac{\text{Opp}}{\text{Adj}} = \frac{1}{\tan(\varphi)}$

Pythagorean Identities

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$1 + \cot^2(\theta) = \csc^2(\theta)$

Double Angle Identities

$\sin(2\varphi) = 2\sin(\varphi)\cos(\varphi)$

$\cos(2\varphi) = \cos^2(\varphi) - \sin^2(\varphi)$

Law of Sines

$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$

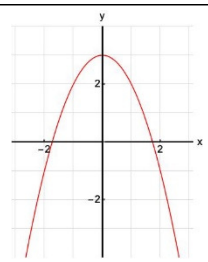
Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$

$b^2 = a^2 + c^2 - 2ac\cos(\beta)$

$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

In exercises 1 – 4, match the equation with graphs a – d below.

a)


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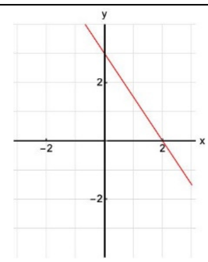

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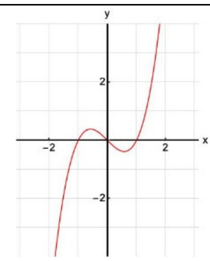
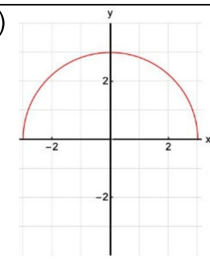


d)


1. $y = -\frac{2}{3}x + 3$
2. $y = \sqrt{9 - x^2}$
3. $y = 3 - x^2$
4. $y = x^3 - x$

Sketch the graph of the equation by point plotting.

5. $y = 0.5x + 2$ | 6. $y = 4 - x^2$ | 7. $y = |x + 2|$ | 8. $y = \sqrt{x} - 6$ | 9. $y = \frac{1}{x+2}$

Find any intercepts of the function.

10. $y = x^2 + x - 2$ | 11. $y^2 = x^3 - 4x$ | 12. $y = x\sqrt{16 - x^2}$ | 13. $y = \frac{x^2 + 3x}{(3x+1)^2}$ | 14. $x^2y - x^2 + 4y = 0$

Sketch the graph of the function. Identify any intercepts and test for symmetry.

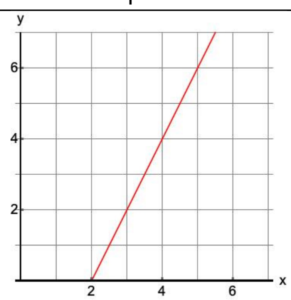
15. $y = (x + 3)^2$ | 16. $y = 2x^2 + x$ | 17. $y = x^3 - 4x$

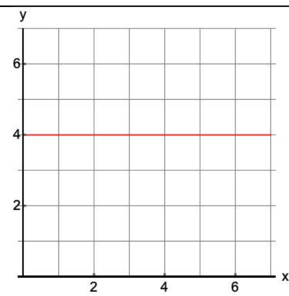
Find the points of intersection of the graphs of the system.

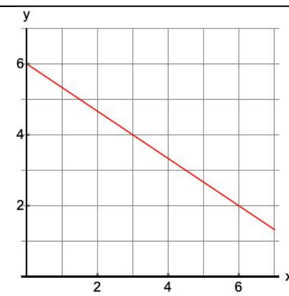
18. $\begin{cases} x + y = 8 \\ 4x - y = 7 \end{cases}$ | 19. $\begin{cases} x^2 + y = 6 \\ x + y = 4 \end{cases}$ | 20. $\begin{cases} x = 3 - y^2 \\ y = x - 1 \end{cases}$

21. Write an equation whose graph has intercepts at x = -4, x = 3, and x = 8.

Estimate the slope of the line from its graph.


22.


23.


24.

25. Sketch the lines through the point (3, 4) with the indicated slopes. Make the sketches on the same set of coordinate axes.
    a. 1 b. -2 c. -¾ d. 0 e. undefined

26. Find the slope of the line through the points $\left(\frac{1}{2}, \frac{2}{3}\right)$ and $\left(\frac{3}{4}, \frac{1}{6}\right)$

Write the equation of the line through the given point with the given slope. Leave your answer in point-slope form.

27. (6, 2) m = 0 | 28. (-4, 3) undefined | 29. (1, 7) m = -3 | 30. (-2, -2) m = 2

31. A moving conveyor is built to rise 1 meter for each 3 meters of horizontal change.
    a. Find the slope of the conveyor.
    b. Suppose the conveyor runs between two floors in a factory. Find the length of the actual conveyor if the vertical distance between floors is 10 feet.

Find the slope and the y-intercept (if possible) of the line.

32. $y = 4x - 3$ | 33. $x + 5y = 20$ | 34. $x = 4$

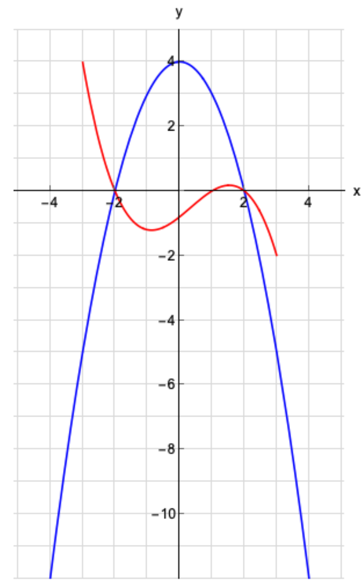
35. You are given the value of a product in 2018 and the rate at which the value of the product is expected to change over the next five years. Write an equation that gives the dollar value (*V*) of the product in terms of the year *t*. (Let *t* = 0 represent the year 2000)
    a. \$1850; \$250 increase per year
    b. \$17,200; \$1,600 decrease per year

36. A company reimburses its sales representatives \$175 per day for lodging and meals plus 48¢ per mile driven. Write a linear equation giving the daily cost (C) to the company in term of the miles driven (m). How much does it cost if a sales representative drives 367 miles on a given day?

37. A real estate office manages an apartment complex with 50 units. When the rent is \$780 per month, all 50 units are occupied. However, when the rent is \$825, the average number of occupied units drops to 47. Assume the relationship between the monthly rent (p) and the number of rented units (x) is linear.
    a. Write a linear equation giving the number of rented units in terms of the rent.
    b. Predict the number of units occupied if the rent is lowered to \$795. Raised to \$894.

38. Find the equation of the line tangent to the circle $x^2 + y^2 = 169$ at the point (5, 12).

39. Use the graphs of f and g, to the right, to answer the following questions:
    a. Identify the domains and ranges of f and g.
    b. Identify $f(-2)$ and $g(3)$.
    c. For what value(s) of x is $f(x) = g(x)$?
    d. Estimate the solution(s) of $f(x) = 2$.
    e. Estimate the solution(s) of $g(x) = 0$.



Evaluate (if possible) the function at the given values. Simplify the results.

40. $f(x) = \sqrt{x+5}$
    a. f(-4)
    b. f(11)
    c. f(-8)
    d. $f(x+\Delta x)$

41. $g(x) = x^2(x-4)$
    a. g(4)
    b. g(3/2)
    c. g(c)
    d. g(t+4)

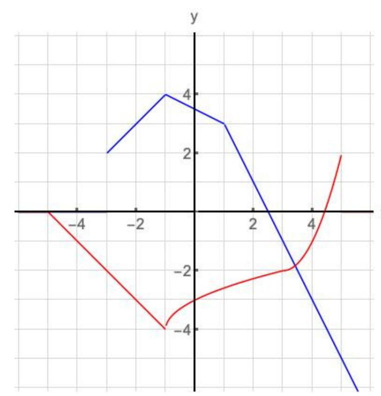
42. Determine the domain and range of $f(x) = \sqrt{x^2 - 3x + 2}$

43. Evaluate $f(x)=\begin{cases} 2x+1, & x<0 \\ 2x+2, & x \geq 0 \end{cases}$ at x = -1, x = 0, x = 2, and $x = t^2 + 1$.

Find the composite functions $(f \circ g)$ and $(g \circ f)$. What is the domain of each composite function? Are the two composite functions equivalent?

44. $f(x) = x^2$, $g(x) = \sqrt{x}$	45. $f(x) = \frac{3}{x}$, $g(x) = x^2 - 1$	46. $f(x) = x^2 - 1$, $g(x) = \cos(x)$	47. $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x+2}$

48. Use the graphs of f and g (below) to evaluate each expression. If the result is undefined, explain why.
    a. $(f \circ g)(3)$
    b. g(f(2))
    c. g(f(5))
    d. $(f \circ g)(-3)$
    e. $(f \circ g)(-1)$
    f. f(g(-1))



49. Sketch a possible graph of the situation.
    a. The speed of an airplane as a function of time during a 5-hour flight.
    b. The height of a baseball as a function of horizontal distance during a homerun.
    c. The amount of a certain brand of sneaker sold by a sporting goods store as a function of the price of the sneaker.

Determine whether the statement is true or false. If it is false, explain why or given an example that shows it is false.

50. If f(a) = f(b), then a = b.
51. A vertical line can intersect the graph of a function at most once.
52. If f(x) = f(-x) for all x in the domain of f, then the graph of f is symmetric with the respect to the y-axis.
53. If f is a function, then f(ax) = af(x)

Factor each expression fully.

54. $-x^2 - x + 12$
55. $-60x^2 + 76x - 24$
56. $3x^3 + 6x^2 - 12x - 24$
57. $x^5 + 8x^4 + 17x^3 - 14x^2 - 84x - 72$
58. $\frac{x^2+8x+16}{x^2-16}$
59. $\frac{3x^3+3x^2-42x-72}{-4x^4+32x^3-52x^2-120x+288}$
60. $\frac{8x^3-125y^3}{8x^3-60x^2y+150xy^2-125y^3}$